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Interpolation by Periodic Radial Basis Functions

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On the unit circle S^1, let d be the natural (geodesic) metric. We investigate the possibility of interpolating arbitrary data on a set of nodes $y_i \in S^1$ by means of a function of the form $x \mapsto \sum_{i=1}^{n} c_i f(d(x, y_i))$. Here f is a function from $[0, \pi]$ to $\mathbb{R}$, and is subject to our choice. The interpolation matrix A having elements $A_{ij} = f(d(y_j, y_i))$ is crucial to this problem. In the basic case, $f(x) = x$, we give necessary and sufficient conditions on the nodes for the invertibility of A. For equally-spaced nodes, we give nearly complete conditions on f for the invertibility of A. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let X be a normed linear space. A function $f: X \to \mathbb{R}$ is said to be *radial* if there exists a function $h: \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) = h(\|x\|)$ for all $x \in X$. A *radial basis* function is any translate of f; that is, a function of the form

$$g(x) = f(x - \xi) = h(\|x - \xi\|),$$

where ξ is any prescribed point of X. A common use of such functions is for interpolation. In this context, one usually has data prescribed at points $\xi_1, \xi_2, ..., \xi_n$ in X, and attempts to interpolate these data by a function of the form

$$x \mapsto \sum_{j=1}^{n} c_j h(\|x - \xi_j\|), \qquad x \in X.$$

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A concrete example may be obtained by setting $X = \mathbb{R}^m$ with the Euclidean norm and $h(t) = (1 + t^2)^{1/2}$. This leads to interpolation by *multiquadrics*, which were introduced by R. Hardy [H] in 1971 for the purpose of representing geological data. There are many other practical examples, and the reader is referred to the surveys by Dyn [Dy] and Powell [P]. Of course, more general settings can easily be imagined. If (X, d) is a metric space, then a radial basis function would have the form

$$g(x) = h(d(x, \xi)), \tag{1.1}$$

where ξ is a prescribed point of X and $h\colon \mathbb{R}_+ \to \mathbb{R}$. Such a generalisation has the following important application. Let $X = S^m$, the unit sphere in $\mathbb{R}^{m+1}$, and let d denote the geodesic distance, given by

$$d(x, y) := \operatorname{Arccos}\langle x, y\rangle.$$

Here $\langle x, y\rangle$ is the usual inner product ("dot" product) of x and y, and the definition adopted for $\cos^{-1} t$ is that

$$\operatorname{Arccos} t := \theta \qquad \text{iff} \quad 0 \leqslant \theta \leqslant \pi \text{ and } \cos\theta = t.$$

The case $m = 2$ is particularly important for the treatment of geophysical data over the whole surface of the earth, for example.

In this paper, we treat the simpler case $X = S^1$ with the geodesic distance function. There are already some marked differences that distinguish this from the case $X = \mathbb{R}$ with its usual absolute value as distance. For example, the interpolation matrix that arises when h is the identity function can be singular in our setting. Some of our results are valid for S^m with $m > 1$, and these instances will be indicated.

It is convenient to define

$$\gamma(t) := \min_{j \in \mathbb{Z}} |t - 2\pi j| \qquad (t \in \mathbb{R}),$$

where $\mathbb{Z}$ is the set of all integers, 0, ± 1, ± 2, ... The following elementary lemma is given without proof; it provides alternative forms for the metric d.

LEMMA 1. *Let* $x_t = (\cos t, \sin t)$. *If* $0 \leqslant s,\ t < 2\pi$, *then*

$$\begin{aligned} d(x_t, x_s) &= \operatorname{Arccos}(1 - \tfrac{1}{2}\|x_t - x_s\|^2) = \min\{|t - s|, 2\pi - |t - s|\} \\ &= \pi - \big|\,|t - s| - \pi\big| = \min_{j \in \mathbb{Z}} |t - s - 2j\pi| = \gamma(t - s). \end{aligned}$$

Henceforth, we identify the points of S^1 with real numbers in the interval $[0, 2\pi)$. We transfer the metric to this interval, where it takes the form

$$d(x, y) = \gamma(x - y) = \min_{j \in \mathbb{Z}} |x - y - 2\pi j|. \tag{1.2}$$

An easy calculation with the aid of Eq. (1.2) verifies that d is a metric on $[0, 2\pi)$.

2. Nodes in General Position

The simplest function h to employ in Eq. (1.1) is obviously the identity, and this section is devoted to this case. We select n distinct points $y_1, y_2, ..., y_n$ in $[0, 2\pi)$; these will be used to define radial basis functions

$$g_i(x) = \gamma(x - y_i) = \min_j |x - y_i - 2\pi j|.$$

We assume that a datum μ_i is prescribed at y_i, for $1 \leqslant i \leqslant n$, and we seek coefficients c_j such that

$$\sum_{j=1}^{n} c_j g_j(y_i) = \mu_i \qquad (1 \leqslant i \leqslant n). \tag{2.1}$$

In order that this problem have a unique solution for any given data μ_i, it is necessary and sufficient that the *interpolation matrix* having elements $d(y_i, y_j)$ be nonsingular.

The results of this section answer completely the question of what sets of nodes lead to a nonsingular interpolation matrix. In addition, we treat certain closely related matters such as the linear independence of the set of functions $g_1, g_2, ..., g_n$. Note that, since S^1 can be embedded in S^m, any result asserting that the interpolation matrix is singular in S^1 is valid in S^m for $m > 1$.

An important observation is that γ is nondifferentiable at every integer multiple of π. It follows that each function g_i is nondifferentiable at all points $y_i + j\pi$ for $j \in \mathbb{Z}$.

Theorem 2. *If the nodes $y_1, y_2, ..., y_n$ are situated in such a way that all the mutual distances $d(y_i, y_j)$ are in the open interval $(0, \pi)$, then the interpolation matrix having elements $d(y_i, y_j)$ is nonsingular.*

Proof. To prove that the matrix is nonsingular, it suffices to show that if a function of the form $f(x) = \sum_{j=1}^{n} c_j d(x, y_j)$ vanishes at each node, then its coefficients c_j must all be zero. Suppose therefore that $f(y_i) = 0$ for $1 \leqslant i \leqslant n$. The elementary identity

$$\gamma(x) + \gamma(x \pm \pi) = \pi \tag{2.2}$$

leads to

$$f(x) + f(x \pm \pi) = \pi \sum_{j=1}^{n} c_j =: q.$$

Since $f(y_i)=0$, we have $f(y_i \pm \pi)=q$. From the nature of γ, we see that f is a continuous piecewise linear function ("spline of degree 1") whose joints ("knots") are the points y_i and $y_i \pm \pi$. By hypothesis, all of these knots are distinct. Hence f' has a jump discontinuity of magnitude $2c_i$ at y_i, for $1 \leqslant i \leqslant n$.

If $q=0$, then f vanishes at all the knots, and we conclude that f is identically zero; hence $f'=0$ and $c_i=0$ for all i. If $q \neq 0$, we can assume by scaling that $q>0$. Then $f(y_i \pm \pi)>0$ for all i. Thus our spline is non-negative. It follows (by considering the graph of f near y_i) that $c_i \geqslant 0$ for all i. If $c_k>0$ for some k, then for any $i \neq k$ we have the contradiction

$$0=f(y_i)=\sum_{j=1}^{n} c_j\, d(y_i, y_j) \geqslant c_k\, d(y_i, y_k)>0. \quad \blacksquare$$

Our next concern is whether the functions g_i form a linearly independent set on $[0, 2\pi)$.

LEMMA 3. *Let $0 \leqslant y_1<y_2< \cdots <y_n<2\pi$. If $\sum_{i=1}^{n} c_i g_i=0$ and if all c_i are nonzero, then n is even (say $n=2k$), $n \geqslant 4$, and $y_{i+k}=y_i+\pi$ for $1 \leqslant i \leqslant k$.*

Proof. First we prove that $y_1<\pi$. If $y_1 \geqslant \pi$, then all the functions $g_2, g_3, \ldots, g_n$ are differentiable at y_1 while g_1 is not differentiable at y_1. This is incompatible with the hypotheses.

Next we prove that for some j, $y_1+\pi=y_j$. If this is not true, then again all $g_2, g_3, \ldots, g_n$ are differentiable at $y_1+\pi$ while g_1 is not. This is contradictory.

The argument just used for y_1 can be applied to each y_i in the interval $[0, \pi)$, and we conclude that if $0 \leqslant y_i<\pi$, then $y_i+\pi$ is also a node. To see that all nodes are thus paired, suppose on the contrary that there exists a node $y_j \in [\pi, 2\pi)$ such that $y_j-\pi$ is not a node. Then g_j is not differentiable at y_j while all other g_i are differentiable. This again is incompatible with our hypotheses.

Now it is clear that n is even, say $n=2k$, and that $y_{i+k}=y_i+\pi$ for $1 \leqslant i \leqslant k$. That $n \geqslant 4$ follows from the observation that when $n=2$ the pair g_1, g_2 is linearly independent. $\blacksquare$

THEOREM 4. *The following conditions are equivalent:*

(i) *$\{g_1, g_2, \ldots, g_n\}$ is linearly dependent;*

(ii) *There exist two nodes y_r and y_s such that $y_r+\pi$ and $y_s+\pi$ are also nodes.*

Proof. Assume that (i) is true. Then there exists a linear dependence of the form

$$\sum_{j=1}^{m} c_j g_{i_j} = 0, \qquad c_j \neq 0,\ i_1 < i_2 < \cdots < i_m,\ y_{i_1} < y_{i_2} < \cdots < y_{i_m}.$$

By applying Lemma 3 to the nodes y_{i_j} we conclude that m is even (say $m = 2k$), that $m \geqslant 4$, and that $y_{i_j} + \pi = y_{i_{j+k}}$ for $1 \leqslant j \leqslant k$. This establishes (ii).

Now assume that (ii) is true. Let $y_r + \pi = y_t$ and $y_s + \pi = y_u$. We employ (2.2) to obtain

$$\pi = \gamma(x - y_r) + \gamma(x - y_r - \pi) = g_r + g_t.$$

It follows that $(g_r + g_t) - (g_s + g_u) = 0$. ∎

THEOREM 5. *Let $y_1, y_2, \ldots, y_n$ be distinct points in $[0, 2\pi)$. Let G be the linear span of the n functions $g_i(x) = d(x, y_i)$. Then* $\dim G = \min(n, n + 1 - k)$, *where*

$$k = \#\{(i,j) : y_i + \pi = y_j\}.$$

Proof. If $y_i + \pi = y_j$ we call (i, j) a "special pair." Thus k is the number of special pairs. Select from $\{1, 2, \ldots, n\}$ a subset I that is maximal with respect to the property of not containing more than one special pair. By relabelling the points, we can assume that $I = \{1, 2, \ldots, m\}$. Now $m = \min(n, n + 1 - k)$. Indeed, if $k = 0$ then $m = n$, while if $k \geqslant 1$ then I can contain one special pair and one element from each of the remaining $k - 1$ special pairs, so that in these cases $m = n + 1 - k$. Since I does not contain two special pairs, Theorem 4 implies that $\{g_1, g_2, \ldots, g_m\}$ is linearly independent.

By the maximality of I, each set $I \cup \{m + i\}$ contains two special pairs. By Theorem 4, each set $\{g_1, g_2, \ldots, g_m, g_{m+i}\}$ is linearly dependent. Hence $\{g_1, g_2, \ldots, g_m\}$ is a maximal linearly independent subset of the original set of functions $\{g_1, g_2, \ldots, g_n\}$. It is therefore a basis for G. ∎

In the remainder of this section we encounter the following interpolation problem:

$$\sum_{j=1}^{n} c_j \, |z_i - z_j| = \mu_i \qquad (1 \leqslant i \leqslant n), \tag{2.3}$$

in which $z_1 < z_2 < \cdots < z_n$ and μ_i are arbitrary real numbers. It is well known that this problem is always solvable. Indeed, the following algorithm solves it:

$$m_i = (\mu_i - \mu_{i-1})/(z_i - z_{i-1}) \qquad (2 \leqslant i \leqslant n)$$

$$m_{n+1} = -m_1 = (\mu_n + \mu_1)/(z_n - z_1)$$

$$c_i = (m_{i+1} - m_i)/2 \qquad (1 \leqslant i \leqslant n).$$

THEOREM 6. *Let $0 = y_1 < y_2 < \cdots < y_n < 2\pi$, and assume that there is exactly one pair (r, s) such that $y_r = y_s + \pi$. Then the matrix $\gamma(y_i - y_j)$ is nonsingular.*

Proof. It suffices to show that for arbitrary data λ_i the interpolation problem

$$\sum_{j=1}^{n} c_j \gamma(y_i - y_j) = \lambda_i \qquad (1 \leqslant i \leqslant n)$$

has a solution. Put $g_i(x) = \gamma(x - y_i)$, and let G be the linear span of $\{g_1, g_2, ..., g_n\}$. Define, for $1 \leqslant i \leqslant n$,

$$y_i' = \begin{cases} y_i + \pi & \text{if } \; y_i < \pi \\ y_i - \pi & \text{if } \; y_i \geqslant \pi. \end{cases}$$

Set $q = \lambda_r + \lambda_s$, where r and s are as in the hypotheses. Define $Z = Y \cup Y'$, where

$$Y = \{y_i : 1 \leqslant i \leqslant n\} \qquad \text{and} \qquad Y' = \{y_i' : 1 \leqslant i \leqslant n\}.$$

By hypothesis, $Y \cap Y' = \{y_s, y_r\}$, and consequently $\# Z = 2n - 2$. Let $Z = \{z_i : 1 \leqslant i \leqslant 2n - 2\}$, where

$$0 = z_1 < z_2 < \cdots < z_{n-1} < \pi = z_n < z_{n+1} < \cdots < z_{2n-2} < 2\pi.$$

Define $\mu_i = \lambda_j$ if $z_i = y_j$ and define $\mu_i = q - \lambda_j$ if $z_i = y_j'$. (Note that there is an index i_0 for which $z_{i_0} = y_r = y_s'$. Hence μ_{i_0} receives two definitions: as λ_r and as $q - \lambda_s$. These two definitions are consistent, by the definition of q.)

By the remarks made just prior to this theorem, there exist coefficients $c_1, c_2, ..., c_n$ for which

$$\sum_{j=1}^{n} c_j |z_i - z_j| = \mu_i$$

for $1 \leqslant i \leqslant n$. It is now asserted that for suitable coefficients b_i we have

$$\sum_{i=1}^{n} c_i |x - z_i| = \sum_{i=1}^{n} b_i \gamma(x - y_i) \qquad (0 \leqslant x \leqslant \pi).$$

To prove this, note first that if $z_i \in Y$ and $z_i = y_j$, then for $0 \leqslant x \leqslant \pi$ we have

$$|x - z_i| = |x - y_j| = \gamma(x - y_j).$$

If $z_i \in Y'$ and $z_i = y_j'$ then for $0 \leqslant x \leqslant \pi$ we have

$$\begin{aligned} |x - z_i| &= |x - y_j'| = \gamma(x - y_j') = \gamma(x - y_j + \pi) = \pi - \gamma(x - y_j) \\ &= \gamma(x - y_s) + \gamma(x - y_r) - \gamma(x - y_j). \end{aligned}$$

Thus, each term $c_i|x - z_i|$ equals a function in G on the interval $[0, \pi]$.

Let $f(x) = \sum_{j=1}^{n} b_j \gamma(x - y_j)$. Then $f \in G$, and it is to be shown that $f(y_i) = \lambda_i$ for $1 \leqslant i \leqslant n$. In fact, we prove the stronger result that $f(z_i) = \mu_i$ for $1 \leqslant i \leqslant 2n - 2$. First, note that

$$\begin{aligned} q = \mu_n - \mu_1 &= f(z_n) - f(z_1) = f(\pi) - f(0) \\ &= \sum_{i=1}^{n} b_i[\gamma(\pi - y_i) - \gamma(y_i)] = \pi \sum_{i=1}^{n} b_i. \end{aligned}$$

Now, if $1 \leqslant i \leqslant n$, then $0 \leqslant z_i \leqslant \pi$ and

$$f(z_i) = \sum_{j=1}^{n} b_j \gamma(z_i - y_j) = \sum_{j=1}^{n} c_j |z_i - z_j| = \mu_i.$$

If $n + 1 \leqslant i \leqslant 2n - 2$, then $z_i = z_k + \pi$ for some $k \in \{1, 2, ..., n\}$. Hence

$$\begin{aligned} f(z_i) &= \sum_{j=1}^{n} b_j \gamma(z_i - y_j) = \sum_{j=1}^{n} b_j \gamma(z_k + \pi - y_j) \\ &= \sum_{j=1}^{n} b_j[\pi - \gamma(z_k - y_j)] = q - \sum_{j=1}^{n} b_j \gamma(z_k - y_j) \\ &= q - f(z_k) = q - \mu_k = \mu_i. \quad \blacksquare \end{aligned}$$

THEOREM 7. *Let $0 = y_1 < y_2 < \cdots < y_n < 2\pi$. These properties are equivalent*:

(i) *the set of n functions $g_i(x) = d(x, y_i)$ is linearly independent*

(ii) *the $n \times n$ matrix $(g_i(y_j))$ is nonsingular.*

Proof. It is obvious that (ii) implies (i). If (i) is true, then by Theorem 5, there can exist at most one pair of nodes such that $y_i = y_j + \pi$. If there is exactly one such pair, then (ii) follows from Theorem 6. If there are no such pairs, then (ii) follows from Theorem 2. $\blacksquare$

THEOREM 8. *Let* $f(x)=ax+b+\sum_{k=1}^{\infty}\alpha_k\cos(2k-1)x$, *with* $\sum_{k=1}^{\infty}|\alpha_k|<\infty$. *Let* $0\leqslant y_1<y_2<\cdots<y_n<2\pi$. *If there are two indices r and s such that* $y_r+\pi$ *and* $y_s+\pi$ *are nodes then the functions* $g_i(x)=f(d(x,y_i))$ *form a dependent set. If* $2b+a\pi=0$, *the same conclusion can be drawn if there is only one index such as r.*

Proof. It is elementary to prove that

$$f(x)+f(\pi-x)=2b+a\pi=:c. \tag{2.4}$$

Now suppose that $y_r+\pi=y_t$. Using Eqs. (2.2) and (2.4), we have

$$\begin{aligned} g_t(x)&=f(d(x,y_t))=f(\gamma(y_t-x))=f(\gamma(y_r+\pi-x))\\ &=f(\pi-\gamma(y_r-x))=c-f(\gamma(y_r-x))=c-g_r(x). \end{aligned}$$

If $c=0$, the dependence sought is $g_t+g_r=0$.

If $c\neq 0$ and if we have also $y_s+\pi=y_v$ then $g_v=c-g_s$ and the promised dependence is

$$(g_t+g_r)-(g_v+g_s)=0. \quad \blacksquare$$

3. EQUALLY-SPACED NODES

If the nodes $y_1, y_2, ..., y_n$ are equally distributed in $[0, 2\pi)$, then a greater degree of generality can be achieved over the results of Section 2. We let

$$y_j=2\pi(j-1)/n \qquad (1\leqslant j\leqslant n),$$

and prescribe a function f: $[0,\pi]\to\mathbb{R}$. One now contemplates the interpolation of data μ_i given at the nodes y_i by a function of the form

$$x\mapsto\sum_{j=1}^{n}c_jf(d(x,y_j)). \tag{3.1}$$

The interpolation matrix now has entries $f(d(y_i,y_j))$, and the problem is to give conditions on f that are necessary and sufficient for the invertibility of this matrix. The nature of this problem requires that f be subject to some further restriction, for we must at least be able to evaluate f at a point in an unambiguous manner. Thus, a space such as $L^1[0,\pi]$ is too large for this purpose. The restriction we choose to make is that f should be continuous, 2π-periodic, and even; further, its cosine series should be absolutely convergent. Expressed more succinctly, we consider only f representable as

$$f(x)=\sum_{k=0}^{\infty}\alpha_k\cos kx, \qquad \sum_{k=0}^{\infty}|\alpha_k|<\infty. \tag{3.2}$$

Since the function values $f(x)$ are (initially) required only for x in the interval $[0, \pi]$, there is no loss of generality in extending f to be even in $[-\pi, \pi]$ and then to be periodic on the whole of the real line.

The reasonableness of our restriction emerges when (3.2) is interpreted as a requirement that the Fourier coefficients of f form a sequence in l^1. Allowing sequences in the space l^2 would, of course, be too lenient, as then f could be an arbitrary element of $L^2[0, 2\pi]$.

LEMMA 9. *If f is even and 2π-periodic, then for x and y in $[0, 2\pi)$ we have $f(d(x, y)) = f(|x-y|) = f(x-y)$.*

Proof. For x and y in $[0, 2\pi)$ the distance function is

$$d(x, y) = \begin{cases} |x-y| & \text{if } \ |x-y| \leqslant \pi \\ 2\pi - |x-y| & \text{if } \ |x-y| > \pi. \end{cases}$$

Thus if $|x-y| \leqslant \pi$ we certainly have $f(d(x, y)) = f(|x-y|)$. If $|x-y| > \pi$ we have

$$f(d(x, y)) = f(2\pi - |x-y|) = f(-|x-y|) = f(|x-y|). \qquad \blacksquare$$

LEMMA 10. *If f is even and 2π-periodic, and if $y_j = 2\pi(j-1)/n$, then the $n \times n$ matrix $A_{ij} = f(d(y_i, y_j))$ is a circulant whose first row is $A_{1,j} = f(y_j)$, $1 \leqslant j \leqslant n$.*

Proof. The definition of a circulant requires that for $j \geqslant i$, $A_{ij} = A_{1,j-i+1}$ while for $j < i$, $A_{ij} = A_{1,j-i+1+n}$. To verify these conditions, first let $j \geqslant i$. Using the preceding lemma, we have $A_{ij} = f(d(y_i, y_j)) = f(|y_i - y_j|) = f(y_j - y_i) = f(y_{j-i+1}) = A_{1,j-i+1}$. Next, let $j < i$. Then $A_{ij} = f(d(y_i, y_j)) = f(|y_i - y_j|) = f(y_i - y_j) = f(y_{i-j+1}) = f(-y_{i-j+1}) = f(2\pi - y_{i-j+1}) = f(y_{n-i+j+1}) = A_{1,j-i+1+n}$. $\blacksquare$

At this juncture, we invoke the following result that can be found implicitly in [A, pp. 123–124], and explicitly in [MM, pp. 65–66; D, pp. 72–73].

LEMMA 11. *If C is an $n \times n$ circulant matrix whose top row is $(\beta_0, \beta_1, ..., \beta_{n-1})$ then the eigenvalues of C are*

$$\lambda_j = \sum_{\nu=0}^{n-1} \beta_\nu e^{2\pi i j \nu/n} \qquad (0 \leqslant j \leqslant n-1).$$

Lemma 11 allows us to compute eigenvalues of the interpolation matrix $A_{ij} = f(d(y_i, y_j))$ when $y_j = 2\pi(j-1)/n$. These results are taken up next.

First, it is convenient to introduce the formalism of discrete Fourier analysis. Two simple definitions are needed:

$$E_k(x) = e^{ikx} \qquad (x \in \mathbb{R}, k \in \mathbb{Z})$$

$$\langle f, g \rangle_n = \frac{1}{n} \sum_{\nu=0}^{n-1} f(2\pi\nu/n)\, \overline{g(2\pi\nu/n)}.$$

LEMMA 12. *Let* $f = \sum_{k=-\infty}^{\infty} \alpha_k E_k$, *with* $\sum_{k=-\infty}^{\infty} |\alpha_k| < \infty$. *(The coefficients* α_k *are permitted to be complex.) Let* A *be the* $n \times n$ *circulant matrix whose top row has the elements* $f(2\pi(j-1)/n)$, *with* $1 \leqslant j \leqslant n$. *Then the eigenvalues of* A *are*

$$\lambda_j = n \sum_{k=-\infty}^{\infty} \alpha_{kn+j} \qquad (1 \leqslant j \leqslant n).$$

Proof. Let $\beta_\nu = f(2\pi\nu/n)$ in Lemma 11 to see that the eigenvalues of A are given by

$$\begin{aligned}
\mu_j &= \sum_{\nu=0}^{n-1} f(2\pi\nu/n)\, e^{2\pi i j\nu/n} = \sum_{\nu=0}^{n-1} f(2\pi\nu/n)\, E_j(2\pi\nu/n) \\
&= n\,\langle f, \bar{E}_j \rangle_n = n \left\langle \sum_{k=-\infty}^{\infty} \alpha_k E_k, \bar{E}_j \right\rangle_n = n \left\langle \sum_{k=-\infty}^{\infty} \alpha_{-k} \bar{E}_k, \bar{E}_j \right\rangle_n \\
&= n \sum_{k=-\infty}^{\infty} \alpha_{-k} \langle E_j, E_k \rangle_n \qquad (0 \leqslant j \leqslant n-1).
\end{aligned}$$

To complete the proof, we use the known fact that $\langle E_j, E_k \rangle_n$ is 1 if $j-k$ is an integer multiple of n, but is 0 otherwise. Using $j - k = rn$, we have

$$\mu_j = n \sum_{r=-\infty}^{\infty} \alpha_{rn-j} \qquad (0 \leqslant j \leqslant n-1).$$

Now let $\lambda_j = \mu_{n-j}$ for $1 \leqslant j \leqslant n$. Then

$$\lambda_j = \mu_{n-j} = n \sum_{r=-\infty}^{\infty} \alpha_{rn-(n-j)} = n \sum_{r=-\infty}^{\infty} \alpha_{(r-1)n+j} = n \sum_{k=-\infty}^{\infty} \alpha_{kn+j}. \quad \blacksquare$$

A special case of Lemma 12 having particular relevance to the interpolation problem will be developed *ab initio*. Since we deal with even periodic functions it is convenient to define

$$C_k(x) = \cos kx \qquad (k \in \mathbb{Z}).$$

Also we define (with n now being fixed)

$$\Delta_k = \begin{cases} 1 & \text{if } n \text{ divides } k \ (k \in \mathbb{Z}) \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 13. *For* $j, k \in \mathbb{Z}$, $\langle C_j, C_k \rangle_n = \frac{1}{2}(\Delta_{k-j} + \Delta_{k+j})$.

Proof. The inner products $\langle C_j, C_k \rangle_n$ and $\langle E_j, E_k \rangle_n$ are real, and consequently

$$\begin{aligned} \langle C_j, C_k \rangle_n &= \langle C_j, \tfrac{1}{2}(E_k + E_{-k}) \rangle_n \\ &= \langle E_j, \tfrac{1}{2}(E_k + E_{-k}) \rangle_n \\ &= \tfrac{1}{2}\langle E_j, E_k \rangle_n + \tfrac{1}{2}\langle E_j, E_{-k} \rangle_n \\ &= \tfrac{1}{2}(\Delta_{j-k} + \Delta_{j+k}). \quad \blacksquare \end{aligned}$$

In subsequent calculations we denote by $\sum'$ a sum in which the first term is halved.

LEMMA 14. *For* $0 \leqslant j \leqslant n-1$,

$$\sum_{k=0}^{\infty}{}' \alpha_k(\Delta_{k+j} + \Delta_{k-j}) = \sum_{k=0}^{\infty} (\alpha_{kn+j} + \alpha_{kn+n-j}).$$

Proof. For $j = 0$, the series on the left reduces to

$$2 \sum_{k=0}^{\infty}{}' \alpha_k \Delta_k = 2 \sum_{r=0}^{\infty}{}' \alpha_{rn}.$$

The series on the right becomes

$$\sum_{k=0}^{\infty} (\alpha_{kn} + \alpha_{(k+1)n}) = \alpha_0 + 2 \sum_{k=1}^{\infty} \alpha_{kn} = 2 \sum_{k=0}^{\infty}{}' \alpha_{kn}.$$

For $1 \leqslant j \leqslant n-1$ we have

$$\begin{aligned} \sum_{k=0}^{\infty}{}' \alpha_k \Delta_{k+j} &= \tfrac{1}{2}\alpha_0 \Delta_j + \sum_{k=1}^{\infty} \alpha_k \Delta_{k+j} = \sum_{k=1}^{\infty} \alpha_k \Delta_{k+j} \\ &= \sum [\alpha_k : k \geqslant 1, k+j = rn] = \sum_{r=1}^{\infty} \alpha_{rn-j}. \end{aligned}$$

Similarly,

$$\sum_{k=0}^{\infty}{}' \alpha_k \Delta_{k-j} = \sum_{r=0}^{\infty} \alpha_{rn+j}.$$

Hence

$$\sum_{k=0}^{\infty}{}' \alpha_k (\Delta_{k+j} + \Delta_{k-j}) = \sum_{r=1}^{\infty} \alpha_{rn-j} + \sum_{r=0}^{\infty} \alpha_{rn+j}$$
$$= \sum_{r=0}^{\infty} (\alpha_{rn+n-j} + \alpha_{rn+j}). \quad \blacksquare$$

THEOREM 15. *Let* $f(x) = \sum'^{\infty}_{k=0} \alpha_k \cos kx$, *with* α_k *real and* $\sum_{k=0}^{\infty} |\alpha_k| < \infty$. *Fixing* n, *let* $y_j = 2\pi(j-1)/n$ *for* $j \in \mathbb{Z}$. *The* $n \times n$ *matrix* $A_{ij} = f(d(y_i, y_j))$ *has eigenvalues given by the equation*

$$\lambda_j = \frac{n}{2} \sum_{k=0}^{\infty} (\alpha_{kn+j} + \alpha_{kn+n-j}) \qquad (0 \leqslant j \leqslant n-1).$$

Proof. By Lemma 10, A is a circulant whose first row is

$$A_{1j} = f(y_j) \qquad (1 \leqslant j \leqslant n).$$

By Lemma 11, the eigenvalues of A are

$$\lambda_j = \sum_{v=0}^{n-1} f(y_{v+1})\, e^{2\pi i j v/n} = n\langle f, \bar{E}_j\rangle_n \qquad (0 \leqslant j \leqslant n-1). \tag{3.3}$$

Since A is real and symmetric, its eigenvalues are real. Hence

$$\lambda_j = n\langle f, C_j\rangle_n \qquad (0 \leqslant j \leqslant n-1).$$

Now substitute the series for f in this equation, obtaining

$$\lambda_j = n \sum_{k=0}^{\infty}{}' \alpha_k \langle C_k, C_j\rangle_n \qquad (0 \leqslant j \leqslant n-1).$$

By Lemma 13, this becomes

$$\lambda_j = \frac{n}{2} \sum_{k=0}^{\infty}{}' \alpha_k (\Delta_{k+j} + \Delta_{k-j}) \qquad (0 \leqslant j \leqslant n-1).$$

Now apply Lemma 14 to this equation to complete the proof. $\blacksquare$

EXAMPLE 16. Consider the function $f(x) = |x|$, defined initially on the interval $[-\pi, \pi]$ and then extended periodically. The Fourier series of f is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}.$$

The coefficient sequence is therefore

$$(\alpha_0, \alpha_1, \alpha_2, ...) = \left(\pi, \frac{-4}{\pi}, 0, \frac{-4}{9\pi}, 0, \frac{-4}{25\pi}, 0, ...\right).$$

By Theorem 15, the eigenvalues of the interpolation matrix $A_{ij} = d(y_i, y_j)$ are

$$\lambda_j = \frac{n}{2} \sum_{k=0}^{\infty} (\alpha_{kn+j} + \alpha_{kn+n-j}) \qquad (0 \leqslant j \leqslant n-1).$$

Let us examine the eigenvalues in greater detail. First, let n be even, $n \geqslant 4$. Let $j \in \{2, 4, 6, ..., n-2\}$. Note that in the series for λ_j we have

$$kn + j \in \{j, j+n, j+2n, ...\} \subset \{2, 4, 6, ...\}$$

and consequently $\alpha_{kn+j} = 0$. In the same way we see that the terms α_{kn+n-j} also vanish. This shows that all eigenvalues $\lambda_2, \lambda_4, \lambda_6, ..., \lambda_{n-2}$ are zero, and that the interpolation matrix A is singular when n is even and greater than 3.

Now let n be odd, $n \geqslant 3$. In this case, by Eq. (3.3) we have

$$\lambda_0 = \sum_{k=0}^{n-1} f(y_{k+1}) = 2 \sum_{k=1}^{(n-1)/2} 2\pi k/n = \frac{\pi}{2n}(n^2 - 1).$$

For $1 \leqslant j \leqslant n-1$, similar calculations, carried out in [XC_1], yield

$$\lambda_j = -\frac{2\pi}{n} \sin^2(mj\pi/n)/\sin^2(j\pi/n),$$

where $n = 2m+1$. In this expression, the denominator $\sin^2 j\pi/n$ is clearly nonzero because $1 \leqslant j \leqslant 2m$. The numerator will be zero only if $mj\pi/n = k\pi$ for some integer k. This leads to $j = k(2 + 1/m)$, and thus k/m must be an integer, r. Then we have $j = rn$. This is not possible, since $1 \leqslant j \leqslant n-1$. Since none of the eigenvalues can be zero, the interpolation matrix is nonsingular. In [XC_1], the cardinal function and other matters are discussed for this case.

THEOREM 17. *Let $y_j = 2\pi(j-1)/n$ for $1 \leqslant j \leqslant n$. The $n \times n$ matrix A having elements $A_{ij} = d(y_i, y_j)$ is singular if and only if n is even. If n is odd, the condition number of A is not worse than $n^2\pi^2$.*

THEOREM 18. *Let* $f(x)=\sum'^{\infty}_{k=0} \alpha_k \cos kx$, *in which* α_k *are real and* $\sum_{k=0}^{\infty} |\alpha_k| < \infty$. *Fix* $n \in \mathbb{N}$. *The following properties of* f *are equivalent*:

(a) *The* $n \times n$ *matrix* $f(d(y_i, y_j))$ *is nonsingular.* (*Here* $y_j = 2\pi(j-1)/n$, $1 \leqslant j \leqslant n$.)

(b) $\prod_{j=0}^{n-1} \sum_{k=0}^{\infty} (\alpha_{kn+j} + \alpha_{kn+n-j}) \neq 0$.

Proof. This follows at once from the computation of the eigenvalues given in Theorem 15. ∎

COROLLARY 19. *Let* f *be as in Theorem* 18. *If the coefficients* α_k *are all strictly positive, then the interpolation matrices* $f(d(y_i, y_j))$ *will be nonsingular for all* n.

The condition on f given in the corollary is obviously far from necessary for the nonsingularity of all interpolation matrices. For example, an f for which $\alpha_k \geqslant 0$ for all k and

$$\alpha_{n^2+j} > 0 \qquad (0 \leqslant j < n;\, n = 1, 2, 3, \ldots)$$

would lead to nonsingular matrices.

If $a \neq 1$, the function

$$f(x) = \frac{1}{1 - 2a\cos x + a^2}$$

satisfies the conditions of Corollary 19. Indeed its cosine coefficients are $\alpha_k = 2a^k(1-a^2)^{-1}$ if $|a| < 1$ and they are $\alpha_k = 2a^{-k}(a^2-1)^{-1}$ if $|a| > 1$ [GR, p. 366]. A similar remark applies to the following functions

$$f(x) = (1 + a\cos x)^{-1} \qquad |a| < 1,$$

$$f(x) = \frac{\cos x}{1 - 2a\cos x + a^2} \qquad a \neq 1.$$

LEMMA 20. *If* $f''(x) \geqslant 0$ *on* $(0, \pi)$ *but is not constantly* 0 *there, then*

$$\text{(a)}\quad 2f'(j\pi - \pi) < (-1)^j \int_{j\pi-\pi}^{j\pi} f(x)\cos x\, dx < 2f'(j\pi)$$

$$\text{(b)}\quad (-1)^j \int_{j\pi-2\pi}^{j\pi} f(x)\cos x\, dx > 0.$$

Proof. Put $P_j = (-1)^j \int_{j\pi-\pi}^{j\pi} f(x)\cos x\, dx$. An integration by parts yields

$$P_j = (-1)^j \left\{ f(x) \sin x \,|_{j\pi-\pi}^{j\pi} - \int_{j\pi-\pi}^{j\pi} f'(x) \sin x \, dx \right\}$$

$$= \int_{j\pi-\pi}^{j\pi} f'(x)(-1)^{j+1} \sin x \, dx$$

$$< f'(j\pi) \int_{j\pi-\pi}^{j\pi} (-1)^{j+1} \sin x \, dx = 2f'(j\pi).$$

The same procedure gives also $P_j > 2f'(j\pi - \pi)$. Writing $F(x) = f(x) \cos x$, we have

$$(-1)^j \int_{j\pi-\pi}^{j\pi} F = (-1)^j \int_{j\pi-\pi}^{j\pi} F + (-1)^j \int_{j\pi-\pi}^{j\pi-\pi} F$$

$$= P_j - P_{j-1} > 2f'(j\pi - \pi) - 2f'(j\pi - \pi) = 0. \quad \blacksquare$$

LEMMA 21. *Let $f \in C^2[0, \pi]$ and satisfy*

(a) $f'(0) \geqslant 0$,

(b) $f''(x) \geqslant 0$ *on* $(0, \pi)$,

(c) f'' *is not constantly* 0 *on* $(0, \pi)$.

Then $(-1)^k \int_0^\pi f(x)\, kx \, dx > 0$ *for* $k = 1, 2, 3, \ldots$.

Proof. Change the variable by replacing x by kx. We then have to prove

$$(-1)^k \int_0^{k\pi} f\left(\frac{x}{k}\right) \cos x \, dx > 0.$$

Denote the integrand in this inequality by $F(x)$. If k is even, write

$$\int_0^{k\pi} F = \int_0^{2\pi} F + \int_{2\pi}^{4\pi} F + \cdots + \int_{(k-2)\pi}^{k\pi} F.$$

Each term on the right in this equation is positive by part (b) of Lemma 20. If k is odd, write

$$\int_0^{k\pi} F = \int_0^{\pi} F + \int_{\pi}^{3\pi} F + \int_{3\pi}^{5\pi} F + \cdots + \int_{(k-2)\pi}^{k\pi} F.$$

The first integral on the right of this equation is negative, by part (a) of Lemma 20. The succeeding terms are also negative, by part (b) of Lemma 20. $\blacksquare$

THEOREM 22. *Let* $y_j = 2\pi(j-1)/n$ *for* $1 \leqslant j \leqslant n$, *and suppose that* f *is an element of* $C^2[0, \pi]$ *such that*

(a) $\int_0^\pi f(x)\,dx > 0$

(b) $f'(0) \geqslant 0$

(c) $f''(x) \geqslant 0$ *on* $(0, \pi)$

(d) f'' *is not identically* 0 *on* $(0, \pi)$.

Then for all even values of n, *the* $n \times n$ *interpolation matrix* $f(d(y_i, y_j))$ *is nonsingular.*

Proof. The cosine series for f has coefficients

$$\alpha_k = \frac{2}{\pi} \int_0^\pi f(x) \cos kx\, dx.$$

We have $\alpha_0 > 0$ by hypothesis (a). For $k = 1, 2, 3, \ldots$, Lemma 21 asserts that $(-1)^k \alpha_k > 0$. By Theorem 15, the eigenvalues of the interpolation matrix are given by

$$\lambda_j = \frac{n}{2} \sum_{k=0}^{\infty} (\alpha_{kn+j} + \alpha_{kn+n-j}) \qquad (0 \leqslant j \leqslant n-1).$$

If n is even, we have

$$(-1)^j \alpha_{kn+j} = (-1)^{kn+j} \alpha_{kn+j} > 0$$
$$(-1)^j \alpha_{kn+n-j} = (-1)^{kn+n-j} \alpha_{kn+n-j} > 0.$$

Hence all terms in the sum defining λ_j have the sign $(-1)^j$. Consequently $(-1)^j \lambda_j > 0$. Since the eigenvalues are nonzero, the matrix is nonsingular. ∎

EXAMPLES 23. The following functions satisfy the hypotheses of Theorem 22.

(1) $(c+x^2)^\beta$, $\beta > \frac{1}{2}$, $c \geqslant 0$

(2) $(c+x^2)^{1/2}$, $c > 0$

(3) $(c+x^2)^\beta$, $0 < \beta < \frac{1}{2}$, $c \geqslant (1-2\beta)\,\pi^2$

(4) $\sum_{j=0}^m c_j x^j$, $c_j \geqslant 0$, $\sum_{j=2}^m |c_j| > 0$, $m \geqslant 2$

(5) $c - \cos(\beta x)$, $c > 0$, $0 < |\beta| \leqslant \frac{1}{2}$

(6) $[q(x)]^\beta$, $\beta \geqslant 1$, $q > 0$, $q' > 0$, $q'' > 0$ on $(0, \pi)$

(7) $\sum_{i=1}^m c_i f_i$, $c_i > 0$, f_i satisfying hypotheses of Theorem 22

(8) $\prod_{i=1}^m f_i, f_i > 0, f_i' > 0, f_i'' > 0$ on $(0, \pi)$.

Note that the function $f(x)=x$, analysed in Example 16, just misses being included in this list. Of course, the eigenvalue behavior in Example 16 is quite different from that observed in the proof of Theorem 22, and in Example 16 it is the odd values of n that give nonsingularity.

4. Discussion

It is possible to exploit the theory of completely monotone functions and positive definite functions to obtain information about interpolation on S^1. Here we sample this interesting aspect of our problem. The fundamental results needed are almost all due to Schoenberg from the era 1938.

Theorem 24. *Let μ be a Borel measure on $[0, \infty)$ such that $\mu(\{0\}) < \mu([0, \infty)) < \infty$. Let*

$$F(t) = \int_0^\infty e^{-s(2-2\cos t)}\, d\mu(s).$$

Then for any distinct points $y_1, y_2, ..., y_n$ in S^d the $n \times n$ matrix $A_{ij} = F(d(y_i, y_j))$ is positive definite and therefore nonsingular.

Proof. Let

$$f(t) = \int_0^\infty e^{-st}\, d\mu(s) \qquad (0 \leqslant t < \infty).$$

By the easy half of the Bernstein–Widder Theorem [B, W_1, W_2], f is completely monotone on $[0, \infty)$. Since the measure μ is, by hypothesis, not concentrated at 0, f is not constant. By a theorem of Schoenberg [S_1], the matrix $f(\|x_i - x_j\|^2)$ is positive definite and hence nonsingular for any n distinct points $x_1, ..., x_n$ in any inner-product space. Applying this to the points $y_j \in S^d$, we have

$$\begin{aligned} f(\| y_i - y_j \|^2) &= f(\| y_i \|^2 + \| y_j \|^2 - 2\langle y_i, y_j \rangle) \\ &= f(2 - 2\cos d(y_i, y_j)) \\ &= F(d(y_i, y_j)). \quad \blacksquare \end{aligned}$$

In [S_2], Schoenberg proved that $\|x_i - x_j\|^\alpha$ is nonsingular for $0 < \alpha < 2$. Hence we can conclude that for distinct $y_1, y_2, ..., y_n$ in S^d, the matrix

$$A_{ij} = (2 - 2\cos d(y_i, y_j))^{\alpha/2}$$

is nonsingular. Similar theorems can be based on Micchelli's work [M]. The following is an example.

THEOREM 25. *Let μ be a Borel measure on $[0, \infty)$ such that*

$$\int_1^\infty t^{-1}\, d\mu(t) < \infty \qquad \textit{and} \qquad \mu(\{0\}) < \mu([0, \infty)) < \infty.$$

Let $F(t) = \int_0^\infty s^{-1}[1 - e^{-s(2-2\cos t)}]\, d\mu(s)$. *Then for any distinct points* $y_1, y_2, ..., y_n$ *in* S^d *the* $n \times n$ *matrix*

$$A_{ij} = F(d(y_i, y_j))$$

is nonsingular.

Proof. By a result essentially due to Schoenberg [S_1], the function f defined by

$$f(t) = \int_0^\infty s^{-1}(1 - e^{-st})\, d\mu(s)$$

has these properties:

(1) $f \in C[0, \infty)$;

(2) $f \in C^\infty(0, \infty)$;

(3) $(-1)^k f^{(k+1)}(t) \geqslant 0$ on $(0, \infty)$, for $k = 0, 1, 2, ...$;

(4) $f(t) > 0$ for $t \in (0, \infty)$;

(5) $f'(t)$ is not constant.

By a theorem of Micchelli, [M], these conditions on f guarantee that for any n distinct points x_i in an inner-product space, the $n \times n$ matrix $f(\|x_i - x_j\|^2)$ is nonsingular. The rest of the proof is just like the proof of Theorem 24. ∎

Schoenberg in [S_3] proved that the most general nonnegative definite function on S^1 is of the form

$$g(t) = \sum_{k=0}^\infty a_k \cos kt, \qquad a_k \geqslant 0, \qquad \sum_{k=0}^\infty a_k < \infty. \tag{4.1}$$

Thus, such a function g has the property that for arbitrary n, $x_i \in S^1$, and $c_i \in \mathbb{R}$,

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j\, g(d(x_i, x_j)) \geqslant 0.$$

In other words, the $n \times n$ matrix $(g(d(x_i, x_j)))$ is nonnegative definite. Problem VII-26 of [PS] asserts that if a matrix B, having elements B_{ij}, is nonnegative definite and does not have a pair of identical rows, then its Schur exponential, having elements $e^{B_{ij}}$, is positive definite.

THEOREM 26. *Interpolation at arbitrary (distinct) nodes $x_1, x_2, ..., x_n$ on S^1 is possible by a function of the form*

$$f(x) = \sum_{j=1}^{n} \lambda_j \exp(g(d(x, x_j))), \qquad x \in S^1$$

provided that g is of the form given in Eq. (4.1), *and has the property that $g(t) \neq g(0)$ for $0 < t \leqslant \pi$.*

Similar results, based upon a knowledge of the *strictly* positive definite functions on S^m, can be given. In this connection, see [XC_2].

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